2.0 DDTV Atmospheric Profiling

2.1 Theory of Operation

The theory of operation for the Difference of Differential-Tilt Variance (DDTV) approach to turbulence profiling was formulated in detail during previous efforts and has been published in open literature [1, 2]. The analysis stems from the more-general theory of covariance of phase aberrations for Kolmogorov turbulence given sources and apertures with arbitrary locations propagating along adjacent (or even overlapping) paths [3]. The primary result from this analysis is that any DDTV statistic generated using an arbitrary array of apertures with an arbitrary array of point sources (at wavelength λ) can be expressed as a weighted integral of $C_n^2(z)$ along the propagation path of length L. Furthermore, DDTV path weighting functions can be evaluated in closed-form given the aperture separation and the source separation normalized to the aperture diameter D for differential-tilt variances which are subtracted when computing the DDTV. Using the analytically-evaluated path weighting functions and the measured DDTV values, the process of C_n^2 profiling involves solving for the unknowns in the underlying system of equations describing the physical scenario. This process is outlined below.

For a given array of apertures and point sources comprising the atmospheric profiler hardware, there will be M-DDTV measurements indexed by the subscript k. The set of DDTV measurements are formulated for the aperture and source geometry of the profiler apparatus, considering only those measurements yielding unique propagation geometries over the path. For each DDTV measurement, the published theory shows that for $k = 1, 2, \ldots, M$:

$$\sigma_{\delta\,k}^2 = \frac{128\sqrt{3}\,\Gamma(8/3)}{\pi^2} \left(\frac{2\pi}{\lambda}\right)^2 D^{5/3} L\, W_{0k} \int_0^1 \mathrm{d}\xi\;\; C_n^2(\xi L)\;\; w_{\delta k}(\xi), \tag{1}$$

where $w_{\delta_k}(\xi)$ is the normalized path-weighting function for $\sigma_{\delta\,k}^2$ and W_{0k} is the normalization factor for each path weighting function. From Eq. (1) we may say that:

$$m_k = \int_0^1 \mathrm{d}\xi\;\; C_n^2(\xi L)\;\; w_{\delta k}(\xi), \tag{2}$$

where m_k is the kth normalized DDTV measurement given by:

$$m_k = \frac{\sigma_{\delta\,k}^2}{\frac{128\sqrt{3}\,\Gamma(8/3)}{\pi^2} \left(\frac{2\pi}{\lambda}\right)^2 D^{5/3} L\, W_{0k}}. \tag{3}$$

Now, suppose that $C_n^2(\xi L)$ is well modeled by N partitions of nearly-uniform turbulence strength such that:

$$C_n^2(\xi L) = \sum_{i=1}^{N} C_{ni}^2 \,\mathrm{rect}\left(\frac{\xi - \xi_i}{l_i}\right), \tag{4}$$

where C_{ni}^2 is the uniform turbulence strength over partition i, ξ_i is the normalized position of the center of the of the ith partition of normalized width l_i (i.e., $\sum_{i=1}^{N} l_i = 1$). With the model for $C_n^2(z)$ given in

Eq. (4), we may use Eq. (2) to write:

$$
\begin{aligned}
m_k &= \int_0^1 \mathrm{d}\xi \sum_{i=1}^{N} C_{ni}^2 \, \mathrm{rect}\left(\frac{\xi - \xi_i}{l_i}\right) w_{\delta k}(\xi) \\
&= \sum_{i=1}^{N} C_{ni}^2 \int_0^1 \mathrm{d}\xi \, \mathrm{rect}\left(\frac{\xi - \xi_i}{l_i}\right) w_{\delta k}(\xi) \\
&= \sum_{i=1}^{N} C_{ni}^2 \int_{\xi_i - l_i/2}^{\xi_i + l_i/2} \mathrm{d}\xi \, w_{\delta k}(\xi) \\
&= \sum_{i=1}^{N} C_{ni}^2 \, p_{ki},
\end{aligned}
\tag{5}
$$

where p_{ki} is integral of the kth weighting function over the ith path partition:

$$
p_{ki} = \int_{\xi_i - l_i/2}^{\xi_i + l_i/2} \mathrm{d}\xi \, w_{\delta k}(\xi). \tag{6}
$$

At this point, we recognize that Eq. (5) represents a system of M equations with N unknowns which may be written compactly in vector-matrix form using the following notation:

$$
\mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_M \end{bmatrix} \qquad \mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ p_{M1} & \cdots & \cdots & p_{MN} \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} C_{n1}^2 \\ C_{n2}^2 \\ \vdots \\ C_{nN}^2 \end{bmatrix}. \tag{7}
$$

Thus, the system of equations specified in Eq. (5) may be rewritten as:

$$
\mathbf{m} = \mathbf{P}\mathbf{c}. \tag{8}
$$

Eq. (8) expresses the transformation of a set of C_n^2 values over the propagation path (a turbulence profile) to the normalized DDTV measurements each of which should represent a unique moment of the turbulence profile.

The goal of any turbulence profiling technique is a problem inverse to the set of equations indicated in Eq. (8); that is, given a set of normalized DDTV measurements, determine the vector of C_n^2 values from which those measurements were derived. If $\mathbf{P}$ were an invertible matrix, then the solution to Eq. (8) would be $\mathbf{c} = \mathbf{P}^{-1}\mathbf{m}$. However, $\mathbf{P}$ is generally not invertible, and we instead seek an optimal solution for Eq. (8). The least-squares estimate $\hat{\mathbf{c}}$ for $\mathbf{c}$ is a common optimal solution [5] given by:

$$
\hat{\mathbf{c}} = \mathbf{H}\mathbf{m}, \tag{9}
$$

where $\mathbf{H}$ is the well-known pseudo-inverse [6] of $\mathbf{P}$ given by:

$$
\mathbf{H} = (\mathbf{P}^T\mathbf{P})^{-1}\mathbf{P}^T. \tag{10}
$$

We will refer to $\mathbf{H}$ as the turbulence profile "reconstructor." Given a set of M normalized DDTV measurements, $\mathbf{H}$ will be used to reconstruct N partitions of C_n^2 over the propagation path. The profile reconstructor is computed directly from the $\mathbf{P}$ matrix, whose elements are derived by integrating the analytic weighting functions of the appropriate DDTV measurements over the desired path partitions.

2.2 Validation of DDTV weighting functions

The DDTV profiling technique is predicated upon the accuracy of the theoretical path weighting functions $w_{\delta_k}(\xi)$ which provide localization of turbulence strength given the measured DDTV values from the profiler apparatus. These weighting functions are independent of the actual turbulence profile and depend only on the geometry of apertures and sources for the profiler hardware. Thus, an essential step in confidently using the theoretical weighting functions is to validate the propagation analysis.

As a means of validating these theoretical weighting functions, we conducted wave-optics simulations of the profiler hardware from which the DDTV measurements could be synthesized. With the wave-optics simulation, we have the ability to inject a single phase screen at discrete locations ξ_i along the path. In the context of Eq. (1), $C_n^2(\xi L) \propto \delta(\xi - \xi_i)$. Hence, the resulting DDTV values measured in the simulation will essentially "sift out" the weighting function w_{δ_k} at each ξ_i. Fig. 2 shows the result of such wave-optics simulations, as compared with the theoretical DDTV path weighting functions for several of the DDTV measurements for the profiler. These results illustrate that the theoretical calculations are accurately capturing the proper path weighting functions for the DDTV turbulence profiling approach.

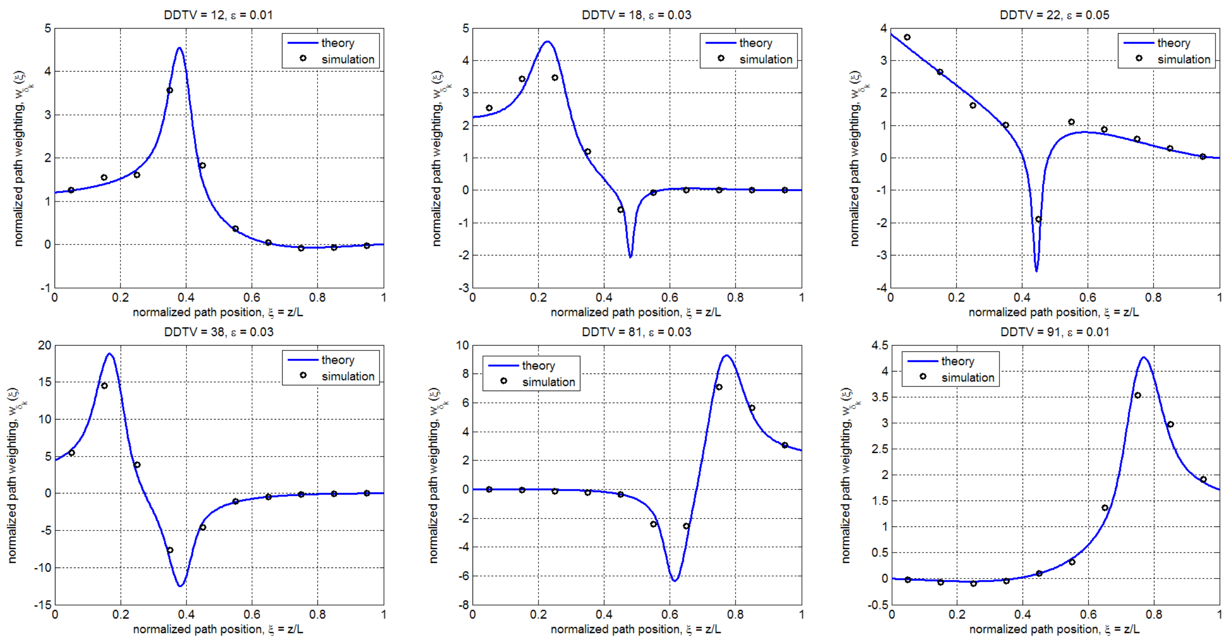

Figure 2: Comparison of theoretical DDTV path weighting functions compared to those measured empirically through wave-optics simulation.

2.3 Estimation algorithm

The general theoretical approach outlined above is used in the context of a broader profile estimation algorithm. A key aspect of this more general algorithm is the application of constraints to the solution resulting from the inversion process to obtain C_n^2 . First, it is recognized that the 3 aperture pairs form 3 separation baselines that can be used to mimic a differential image motion monitor (DIMM) or "r_0 meter." In this approach, we can use tilt (spot centroid) data directly from the profiler units with standard techniques for r_0 estimation. Given the estimated tilt variance σ_T^2, we can compute:

$$r_0 = \left(\frac{\sigma_T^2 D^{1/3}}{0.1816\ \lambda^2} \right)^{-3/5} \tag{11}$$

Since this estimate of r_0 may be corrupted by telescope mount jitter, we instead use differential-tilt variance for 3 aperture pairs on each unit to form individual estimates of r_0. For the Gen-2 Long Range Profiler, the 2 short-baseline aperture pairs with separation d_s have a normalized separation of $d_s/D = 1.61$ and the single long-baseline aperture pair has a normalized separation of $d_l/D = 7.09$, where D is the subaperture diameter. Fig. 3 shows the differential tilt variance $\sigma_{\Delta T}^2$ normalized to the asymptotic value $(2\sigma_T^2)$ as a function of d/D. The 2 aperture-pair baseline values for the Gen-2 system are indicated on the plot, along with the fraction $c_{X,Y}(d/D)$ of the asymptotic value expected for each baseline for X and Y tilts, respectively. Using the values shown in Fig. 3, the r_0 values are related to

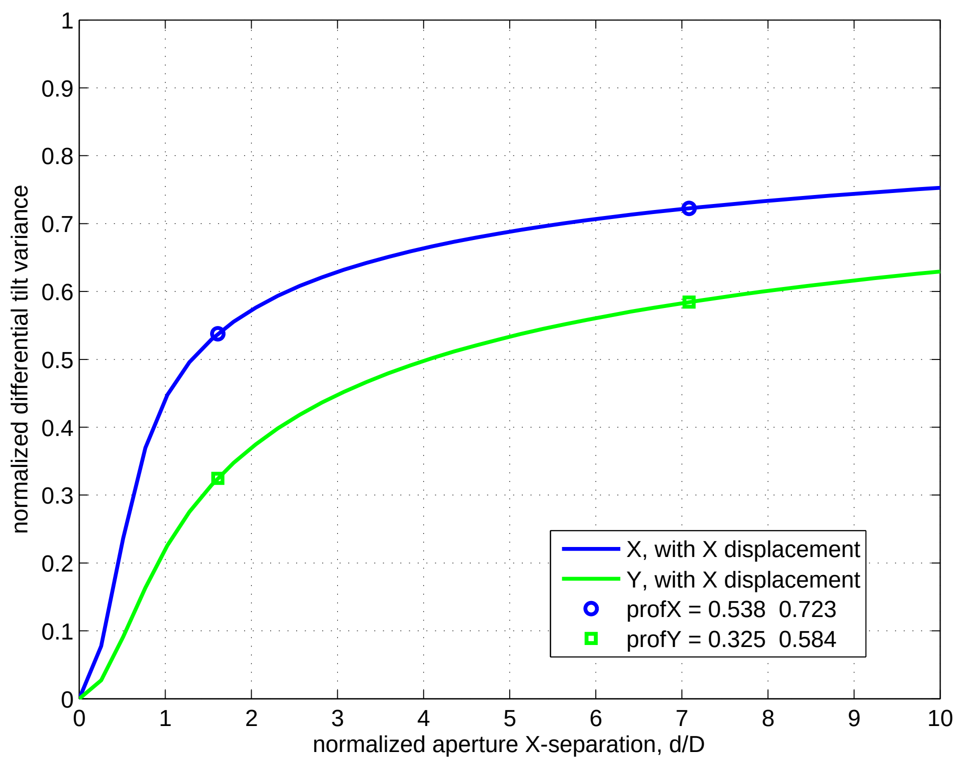

Figure 3: Normalized differential tilt variance given aperture separation for point-sources as determined from tilt covariance theory. Relative aperture separation d/D for the aperture baselines supported the profiler faceplate.

the differential-tilt variance as

$$\text{short baseline: } r0_{X,Y}^{-5/3} = D^{1/3}\sigma_{\Delta T_{X,Y}}^2 \left[2c_{X,Y}(d_s/D) \times 0.1816\ \lambda^2\right]^{-1}, \tag{12}$$

$$\text{long baseline: } r0_{X,Y}^{-5/3} = D^{1/3}\sigma_{\Delta T_{X,Y}}^2 \left[2c_{X,Y}(d_l/D) \times 0.1816\ \lambda^2\right]^{-1}, \tag{13}$$

from which r_0 can be computed given $\sigma_{\Delta T_{X,Y}}^2$.

Estimates of $r_{0_{A,B}}$ values measured from either end of propagation path designated A and B permits constraints to be introduced to inversion. Knowing these propagation parameters provides considerable information about the turbulence strength over the path prior to determining the profile. The primary inversion method that we have used in solving Eq. (8) for $C_n^2(z)$ is called Stochastic Parallel Gradient Descent (SPGD) [4]. In this method, an initial estimate of the turbulence profile is made by equating $r_{0_{A,B}}$ value to a uniform $C^2_{n_{A,B}}$ value over the path that would result in the measured $r_{0_{A,B}}$, and assuming a linear variation of C_n^2 over the path. From there, the SPGD applies iterative perturbations to the $C_n^2(z)$ solution, checking the value of an optimization metric at each step. The optimization metric is computed as the weighted sum of errors between the measured DDTV values and the theoretical DDTV values given the current solution as well as the errors in the measured versus theoretical $r_{0_{A,B}}$. In the process, a larger weight is given to the DDTV measurements than to the atmospheric parameters. The SPGD optimization process continues until an error goal (relative to initial estimate) is reached, or until a specified maximum number of iterations is reached. The reported turbulence profile is that which minimizes the relative error.

The steps involved in the turbulence profile estimation process using the raw centroids, DDTV values, $r_{0_{A,B}}$ measurements, and the SPGD optimization are summarized below:

1. Compute DDTV values from centroid data having passed image quality checks (SNR, saturation, spot-clipping)
   - 6 centroids from each image, X and Y axis, A and B profiler unit
   - Uses “reconstructor” file to determine which DDTV values are computed
2. Compute r_0 values treating profiler data as ensemble of DIMMs
   - 3 r_0 values from each side corresponding to 3 aperture-pair baselines
3. Compute a C_n^2 profile for each combination of r_{0_A} and r_{0_B}
   - Total of 9 $r_{0_{A,B}}$ combinations
   - Apply SPGD inversion for DDTV measurements and r_0 values from each side
   - Average profile estimates for each of the $r_{0_{A,B}}$ combinations
4. Apply to X measurements and Y measurements separately, then average C_n^2 profiles to produce “best estimate”
   - Full ensemble of C_n^2 estimates are output for diagnostic purposes

Once the final turbulence profile is output from a set of data, we use these C_n^2 values as input to various functions in our `ATMTools` toolbox for MATLAB to compute propagation parameters for the atmospheric path. One of outputs computed from the turbulence profile is the value of $r_{0_{A,B}}$. These can be compared to the values reported from processing the profiler data as an ensemble of DIMMs in order to determine the consistency of the profile results with the alternative (profile-independent) propagation parameters.